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Positivity and convergence in fermionic quantum field theory

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Abstract

We derive norm bounds that imply the convergence of perturbation theory in fermionic quantum field theory if the propagator is summable and has a finite Gram constant. These bounds are sufficient for an application in renormalization group studies. Our proof is conceptually simple and technically elementary; it clarifies how the applicability of Gram bounds with uniform constants is related to positivity properties of matrices associated to the procedure of taking connected parts of Gaussian convolutions. This positivity is preserved in the decouplings that also preserve stability in the case of two–body interactions.

1 Introduction

In fermionic field theories with an infrared and an ultraviolet cutoff, perturbation theory converges. Perturbation theory in bosonic theories always diverges. When representing the perturbation series in terms of Feynman graphs, this is often stated in the way that, although there are as many Feynman graphs contributing to the fermionic perturbation expansion as to the bosonic one, there are sign cancellations due to the fermionic antisymmetry that lead to convergence. This explanation is correct, but it is another matter to make the cancellations explicit in a way that one can use them to remove the cutoffs in nonperturbative constructions of fermionic models. This was done for the Gross–Neveu model in [1] and [2]. Recently, there have been various results that make these arguments more explicit [6, 7] and further simplify them, in particular by avoiding cluster–expansion type arguments [3, 4, 5]. Our results go in the same direction, but we believe that they provide an additional structural understanding of how the sign cancellations occur and further reduce the technicalities in the proofs. In our proof, we do not need to expand in Feynman graphs or deal with combinatorial identities that go beyond the most standard tree graph techniques.

The basic reason for convergence is that correlation functions of fermions are determinants of a matrix whose entries are given by the covariance. In contrast to the permanents of bosonic theory, which grow factorially, such determinants are usually bounded by a constant to the degree of the monomial that is integrated. The "usually" is made more precise in the case where Gram bounds apply.

It would be overly optimistic to assume that a *connected* correlation function can also be represented as a determinant. This is because a determinant is a sum over permutations, and permutations decompose the index set into partitions (the cycles). The sum over all permutations always contains some where the vertex structure of the graph does not connect all different cycles, and then the associated graph in the Feynman graph expansion is not connected. Thus the connected correlation functions, which receive contributions only from connected graphs, cannot simply be determinants. However, if one takes out a minimal connected subgraph, namely a tree, and sums over graphs that contain this tree, one can hope to get back a determinant. Moreover, because the number of tree graphs is much smaller than that of all graphs, a tree sum representation is a good starting point for a convergence proof.

One simple way to do the resummation, which we shall describe below because it was at the beginning of this work, is a resummation of the Mayer

graphs for the generating functional of the connected Green functions in terms of trees, in the way proposed by Penrose a long time ago [11]. This provides a resummation of Feynman graphs and thus clarifies which graphs get combined to determinants. But although a Gram bound applies to every term in the sum over trees, the Gram constant depends on the tree and we have no uniform bounds for it yet (although they may be possible). The investigation for the reasons for this problem led us in a natural way to a positivity condition which implies uniformity, and in fact, optimal Gram constants.

A priori, positivity plays no role in the definition of fermionic theories (unlike the bosonic case, where only positive covariances give well–defined Gaussian measures). In particular, fermionic covariances of physically interesting models do not have any positivity properties. Thus it may appear surprising that a positivity condition plays a role in such theories. What is really required is, however, not the positivity of the covariance but that of a connectivity matrix that appears in the tree representation of the connected correlation functions. It turns out that this positivity condition is precisely equivalent to the condition for two–body interactions that stability be preserved in the decoupling expansion for the connected correlation functions.

This explains, at least partially, why the applicability of Gram bounds is fragile in that it is usually destroyed by expansion methods that are not arranged carefully, and it also implies that a Brydges–Battle–Federbush (BBF) representation, which preserves stability and hence positivity, leads to uniform Gram estimates. Indeed, it is optimal in that the Gram constant simply remains the same as before.

We apply the Gram estimates to show norm bounds that are simple but strong enough to study renormalization group flows and to construct fermionic models nonperturbatively.

After finishing our proof, we discovered that Lesniewski [12] had found an explicit Gram representation for the BBF decoupling and used it to prove cumulant bounds. The advantage of our method is that the positivity condition makes it obvious why the Gram bounds work by a nontechnical argument, and thus provides a conceptually and technically simple proof, in which details of explicit representations (such as Lesniewski's Gram representation, which almost appears as a miracle at first sight) are not needed.

In Sect. 2, we give the precise setup and state the main analyticity theorem. Sect. 3 contains its proof and the formulas for the connected correlation functions. In Sect. 4, we discuss the results and some relations to other approaches.

We have written this paper so that it should be understandable for non–experts. We assume only that the reader is familiar with operations on finite–dimensional Grassmann algebras and some elementary notions of graph theory, as well as the basic connectedness formulas of polymer expansions. All these prerequisites are well–documented, see e.g. [8, 9, 10] for polymer expansions and Appendix B of [10] for Grassmann algebras.

2 The setup and the main result

2.1 The Gram Bound

To do combinatorics with generating functions, it is convenient to introduce a discretization, i.e. a space–time lattice even in the theory with a cutoff (we shall follow the conventions of [10]). Our formulas allow us to take the continuum limit (at fixed momentum space cutoff) in our representation of the connected correlation functions. For the estimates, it does not make any difference if the lattice is kept or not.

For the moment, we just assume that the Grassmann algebra is generated by fermionic fields $\psi(X), \bar\psi(X)$, where $X \in \mathbb{X}$ and $\mathbb{X}$ is a finite set. A typical example for $\mathbb{X}$ is a lattice times a set of colour and spin indices. We write sums over $\mathbb{X}$ (possibly times a scaling factor, such as a power of a lattice spacing) as $\int_{\mathbb{X}} \mathrm{d}X$ or briefly $\int \mathrm{d}X$ and denote by $\delta_{\mathbb{X}}(X,Y)$ the Kronecker delta on $\mathbb{X}$, scaled such that $\int \mathrm{d}X\, f(X)\delta_{\mathbb{X}}(X,Y) = f(Y)$. The Grassmann derivatives $\frac{\delta}{\delta\psi(X)}$ anticommute and are normalized such that $\frac{\delta}{\delta\psi(X)}\psi(Y) = \delta_{\mathbb{X}}(X,Y)$. For another family of Grassmann variables $\eta(X)$, we denote $(\psi,\eta) = \int \mathrm{d}X \psi(X)\eta(X)$; by the Grassmann nature of the fields, $(\psi,\eta) = -(\eta,\psi)$. To a fermionic bilinear form

$$(\bar\psi, C\psi) = \int \mathrm{d}x\mathrm{d}y\; \bar\psi(x)\, C(x,y)\, \psi(y) \tag{1}$$

we associate a Gaussian expectation value on the Grassmann algebra by defining

$$\left\langle \mathrm{e}^{(\overline{\eta},\psi)+(\bar\psi,\eta)} \right\rangle = \int \mathrm{d}\mu_C(\psi)\; \mathrm{e}^{(\overline{\eta},\psi)+(\bar\psi,\eta)} = \mathrm{e}^{(\overline{\eta},C\eta)} \tag{2}$$

(the source terms being Grassmann variables as well). The elements of the Grassmann algebra are polynomials

$$V(\psi) = \sum_{m,\overline{m}\ge 0} \int \mathrm{d}^{\overline{m}}\underline{X}\mathrm{d}^{m}\underline{X}' v_{\overline{m},m}(\underline{X},\underline{X}')\bar\psi^{\overline{m}}(\underline{X})\psi^{m}(\underline{X}') \tag{3}$$

where $\underline{X} = (X_1, \dots X_m)$ and $\psi^m(\underline{X}) = \psi(X_1)\dots\psi(X_m)$. The sums over m and $\overline{m}$ are finite sums because of the nilpotency of the Grassmann variables. The coefficient function is chosen antisymmetric under permutation of the X variables and antisymmetric under permutation of the X' variables because any other part of it would cancel out in (3). We call V even if $v_{\overline{m},m} = 0$ unless $m + \overline{m}$ is even (this is in particular the case if $v_{\overline{m},m} = 0$ unless $m = \overline{m}$, but we do not need this more special condition here). If V is even, it commutes with all other elements of the Grassmann algebra. Here and in the following, the notation $V(\psi)$ means that V is a polynomial in ψ and $\bar{\psi}$ (similarly, $\int \mathrm{d}\mu_C(\psi)$ also involves integration with respect to $\bar{\psi}$).

The basic reason for the convergence of fermionic perturbation expansions is the fermionic antisymmetry. The Gaussian integral of a monomial is zero unless there are as many $\bar{\psi}$ as ψ in it. In that case, it is the determinant

$$\left\langle \prod_{l=1}^{p} \bar{\psi}(Y_l) \prod_{k=1}^{p} \psi(X_k) \right\rangle = (-1)^{p(p+1)/2} \det \left(C(X_k, Y_l) \right)_{k,l} \tag{4}$$

We assume that the propagator can be written as an inner product on some Hilbert space $\mathcal{H}$, that is,

$$\begin{array}{ll} & \forall X \in \mathbb{X}\ \exists f_X, g_X \in \mathcal{H}: \quad C(X,Y) = \langle f_X\ ,\ g_Y \rangle, \\ \text{and} & \exists \gamma_C > 0\ \forall X \in \mathbb{X}: \|f_X\| \le \gamma_C,\ \|g_X\| \le \gamma_C. \end{array} \tag{5}$$

Then the *Gram bound* for the determinant (see, e.g., [10], Appendix B.4). implies

$$\left| \left\langle \prod_{l=1}^{p} \bar{\psi}(Y_l) \prod_{k=1}^{p} \psi(X_k) \right\rangle \right| \le {\gamma_C}^{2p}. \tag{6}$$

For models of quantum field theory, a representation (5) typically holds with a finite Gram constant γ_C if cutoffs are present.

2.2 Gaussian convolutions

For $h > 0$, we define the seminorm $\|V\|_h$ of an element of the Grassmann algebra given by (3) by

$$\|V\|_h = \sum_{\substack{m,\overline{m} \ge 0 \\ m+\overline{m} \ge 1}} |v_{\overline{m},m}| h^{\overline{m}+m} \tag{7}$$

where $|v_{\overline{m},m}|$ is the standard norm

$$|v_{\overline{m},m}| = \max_{i \in \mathbb{N}_{\overline{m}+m}} \sup_{X_i} \int \prod_{j \ne i} \mathrm{d}X_j\ |v_{\overline{m},m}(X_1, \dots, X_{\overline{m}+m})|. \tag{8}$$

We do not assume translation invariance. We assume that the norm of C is finite: there is a constant ω_C such that

$$\begin{aligned} |C| &= \max\left\{\sup_X \int |C(X,Y)|\mathrm{d}Y \;,\; \sup_X \int |C(Y,X)|)\mathrm{d}Y\right\} \\ &\le \omega_C {\gamma_C}^2. \end{aligned} \tag{9}$$

On the full Grassmann algebra, $\|\cdot\|_h$ is only a seminorm because the term $\overline{m} = m = 0$ is left out in (7) and thus all constant polynomials K have $\|K\|_h = 0$. On the subspace of Grassmann polynomials with field–independent term equal to zero, $\|\cdot\|_h$ is a norm. The effective action $W(V)$ defined below is defined such that $W(V)(0) = 0$, so it is in that subspace.

Let

$$U(\phi) = (\mu_C * V)(\phi) = \int \mathrm{d}\mu_C(\psi) V(\psi + \phi) \tag{10}$$

be the convolution of V with μ_C. U has an expansion of type (3), with

$$\begin{aligned} u_{\overline{m},m}(\underline{Y},\underline{Y}') &= \sum_{\bar n \ge \overline{m}} \tbinom{\bar n}{\overline{m}} \sum_{n\ge m} \tbinom{n}{m} \int \mathrm{d}^{\bar n - \overline{m}} \underline{X} \int \mathrm{d}^{n-m}\underline{X} \\ & \quad (-1)^{m(\bar n - \overline{m})} v_{\bar n, n}(\underline{Y},\underline{X},\underline{Y}',\underline{X}') \left\langle \bar\psi^{\bar n - \overline{m}}(\underline{X}) \psi^{n-m}(\underline{X}')\right\rangle . \end{aligned} \tag{11}$$

Here we used the antisymmetry of the coefficient $v_{\overline{m},m}$. Taking the norm gives, by the Gram estimate (6),

$$|u_{\overline{m},m}(\underline{Y},\underline{Y}')| \le \sum_{n\ge m} \binom{n}{m} \sum_{\bar n \ge \overline{m}} \binom{\bar n}{\overline{m}} |v_{\bar n, n}| {\gamma_C}^{n-m+\bar n - \overline{m}}, \tag{12}$$

so

$$\|\mu_C * V\|_h \le \|V\|_{h+\gamma_C}. \tag{13}$$

Thus integrating over fermionic variables only shifts the norm parameter by the Gram constant. This is in strong contrast to bosonic problems, and a first indication for the convergence of perturbation theory for fermions. The above estimate is, however, not sufficient because it does not lead to bounds that are uniform in $|\mathbb{X}|$. For this we need to assume decay of the covariance and of the v_m, and consider connected functions, such as, e.g., generated by the effective action.

2.3 A norm bound for the effective action

Let V be even. We define the effective action as

$$W(V)(\psi) = \log \frac{1}{Z} \int \mathrm{d}\mu_C(\psi')\, \mathrm{e}^{V(\psi+\psi')} = \left(\log \frac{1}{Z} \mu_C * \mathrm{e}^V\right)(\psi) \tag{14}$$

where $Z = \mu_C * e^V|_{\psi=0}$, so that $W(V)(0) = 0$. For finite $\mathbb{X}$, the argument of the logarithm, $\frac{1}{Z}\mu_C * e^V$, is a polynomial whose constant term is 1; thus $W(\psi)$ is well-defined if $\|V\|_h$ is small enough (depending on $\mathbb{X}$) because the expansion for the logarithm terminates after a finite number of terms by nilpotency of the Grassmann variables. The following theorem implies that analyticity holds uniformly in $|\mathbb{X}|$ provided the Gram constant γ_C and the decay constant ω_C are bounded uniformly in $|\mathbb{X}|$.

Theorem 1 *Assume* (5) *and* (9). *Let V be even, $V(0) = 0$, and $h' = h + 3\gamma_C$. If $\omega_C \|V\|_{h'} < 1$ then W is analytic in V and in the fields, and*

$$\|W\|_h \leq -\frac{1}{\omega_C} \log\left(1 - \omega_C \|V\|_{h'}\right). \tag{15}$$

Let $W(V) = \sum_{p\geq 1} W_p(V)/p!$ be the expansion of W in powers of V. Then for all $P \geq 1$,

$$\left\| W(V) - \sum_{p=1}^{P} \frac{1}{p!} W_p(V) \right\|_h \leq {\omega_C}^P \frac{{\|V\|_{h'}}^{P+1}}{1 - \omega_C \|V\|_{h'}}. \tag{16}$$

Moreover, we can replace h' by $h'' = h + 2\gamma_C$ in (15) *and* (16) *if we replace ω_C by $2\omega_C$ in these bounds.*

In particular, for $P = 1$,

$$\|W(V) - \mu_C * V\|_h \leq \omega_C \frac{{\|V\|_{h'}}^2}{1 - \omega_C \|V\|_{h'}}. \tag{17}$$

A difference to the linear estimate (13) is that the shift in the norm parameter h is not γ_C but $\beta\gamma_C$ with $\beta > 1$. It will be explicit in the proof where this factor comes from; the last statement of the theorem shows that there is some freedom in moving factors around in the constants. However, we have not been able to prove a bound where the norm parameter shifts only by γ_C.

We shall discuss a related bound in Section 3.6.

3 The expansion for the effective action

3.1 Connected parts and logarithms

In this section, we briefly recall a characterization of connected parts and their role in taking logarithms. Let $\mathbb{N}_p = \{1, \ldots, p\}$, let $\mathcal{A}$ be a commutative

algebra with unit 1, and assume a function

$$\alpha : \mathbb{P}(\mathbb{N}_p) \to \mathcal{A}, \quad Q \mapsto \alpha(Q) \tag{18}$$

with $\alpha(\emptyset) = 1$ to be given (here $\mathbb{P}(M)$ is the power set of M).

Lemma 1 *There is a unique function*

$$\alpha_{\rm c} : \mathbb{P}(\mathbb{N}_p) \to \mathcal{A}, \quad Q \mapsto \alpha_{\rm c}(Q), \quad \alpha_{\rm c}(\emptyset) = 0, \tag{19}$$

that satisfies

$$\forall Q \subset \mathbb{N}_p : \alpha(Q) = \sum_{\substack{J_0 \subset Q \\ \min Q \in J_0}} \alpha_{\rm c}(J_0)\, \alpha(Q \setminus J_0). \tag{20}$$

Moreover, $\alpha(Q)$ is the sum over partitions of Q of products of $\alpha_{\rm c}$ of the elements of the partition (here $\dot\cup$ denotes the disjoint union):

$$\alpha(Q) = \sum_{m \ge 1} \frac{1}{m!} \sum_{\substack{I_1, \ldots I_m \ne \emptyset \\ I_1 \dot\cup \ldots \dot\cup I_m = Q}} \prod_{l=1}^{m} \alpha_{\rm c}(I_l) \tag{21}$$

Proof: Induction on $|Q|$ gives existence and uniqueness of $\alpha_{\rm c}$: For $|Q| = 1$, (20) is simply $\alpha_{\rm c}(Q) = \alpha(Q)$. Once $\alpha_{\rm c}(Q')$ has been determined for all Q' with $|Q'| < |Q|$, (20) is solved in the form

$$\alpha_{\rm c}(Q) = \alpha(Q) - \sum_{\substack{J_0 \subset Q \\ \min Q \in J_0 \ne Q}} \alpha_{\rm c}(J_0)\, \alpha(Q \setminus J_0) \tag{22}$$

The right hand side of (21) solves (20). ■

The convention $\alpha_{\rm c}(\emptyset) = 0$ has no consequences because $\alpha_{\rm c}(\emptyset)$ never appears in any formula.

Lemma 2 *As a formal series in $\alpha_{\rm c}$,*

$$\log\,(1 + \sum_{\substack{Q \subset \mathbb{N}_p \\ Q \ne \emptyset}} \alpha(Q)) = \sum_{m \ge 1} \frac{1}{m!} \sum_{\substack{I_1, \ldots I_m \subset \mathbb{N}_p \\ \text{all nonempty}}} \mathcal{U}_{\rm c}^{(m)}(I_1, ..., I_m) \prod_{l=1}^{m} \alpha_{\rm c}(I_l) \tag{23}$$

where $\mathcal{U}_{\rm c}^{(1)} = 1$ and for $m \ge 2$,

$$\mathcal{U}_{\rm c}^{(m)}(I_1, \ldots, I_m) = \sum_{G \in \mathcal{G}_{\rm c}(\mathbb{N}_m)} \prod_{(i,j) \in G} \gamma(I_i, I_j) \tag{24}$$

with $\gamma(I_i, I_j) = -1$ *if* $I_i \cap I_j \neq \emptyset$ *and* 0 *otherwise, and* $\mathcal{G}_{\rm c}(\mathbb{N}_p)$ *the set of connected graphs on* $\mathbb{N}_p$*. In particular, if* $\lambda_1, \ldots, \lambda_p$ *are formal parameters, then*

$$\frac{\partial^p}{\partial\lambda_1 \ldots \partial\lambda_p} \log\,(1 + \sum_{\substack{Q \subset \mathbb{N}_p \\ Q \neq \emptyset}} \alpha(Q) \prod_{q \in Q} \lambda_q)\Big|_{\lambda=0} = \alpha_{\rm c}(\mathbb{N}_p). \tag{25}$$

Proof: When (21) is inserted to replace $\alpha(Q)$, $\zeta = 1 + \sum \alpha(Q)$ takes the form of a polymer partition function, with the nonempty subsets of $\mathbb{N}_p$ as polymers and disjointness as the compatibility relation. Eq. (23) is the standard polymer formula for the logarithm of the partition function [8, 10]. Eq. (25) follows by noting that for all $m \geq 2$, the connectedness condition in the function $\mathcal{U}_{\rm c}^{(m)}$ implies that after differentiation, some factors λ_i remain, so that evaluating at zero picks out the term $m = 1$ from the sum (see [10], Section 2.5). ■

3.2 Connected parts of the Laplacian

We expand the effective action

$$W(\lambda V) = \sum_{p \geq 1} \frac{\lambda^p}{p!} W_p(V) \tag{26}$$

with $W_p(V) = \langle V; \ldots; V \rangle - \frac{\partial^p}{\partial\lambda^p} \log Z\,|_{\lambda=0}$, where, for elements $V_1, \ldots, V_p$ of the even subalgebra,

$$\langle V; \ldots; V \rangle = \left[\tfrac{\partial^p}{\partial\lambda_1 \ldots \partial\lambda_p} \log \left(\mu_C * {\rm e}^{\lambda_1 V_1 + \ldots + \lambda_p V_p} \right) \right]_{\lambda_q = 0\ \forall q} \tag{27}$$

is the connected correlation function of $V_1, \ldots, V_p$. It is an element of the even subalgebra. The subtraction of $\log Z$ removes the ψ–independent term from $W_p(V)$. Because the derivative is evaluated at $\lambda = 0$, we can replace $\mu_C * {\rm e}^{\lambda_1 V_1 + \ldots + \lambda_p V_p}$ by

$$\mu_C * \prod_{q=1}^{p} (1 + \lambda_q V_q) = 1 + \sum_{\substack{Q \subset \mathbb{N}_p \\ Q \neq \emptyset}} \alpha(Q) \prod_{q \in Q} \lambda_q \tag{28}$$

with

$$\alpha(Q) = \mu_C * \prod_{q \in Q} V_q. \tag{29}$$

Similarly, we can replace Z by (28) evaluated at $\psi = 0$. Because all V_q are in the even subalgebra, $\alpha(Q)$ is in the even subalgebra, and hence all α's commute. Thus, by Lemma 2,

$$\langle V; \ldots; V \rangle = \alpha_{\mathrm{c}}(\mathbb{N}_p). \tag{30}$$

We now rewrite Gaussian convolutions in terms of the action of a Laplacian acting on p independent copies of the field ψ; this is convenient for doing the combinatorics.

Lemma 3 *Let* $\Delta = \sum\limits_{q,q'=1}^{p} \Delta_{qq'}$ *where*

$$\Delta_{qq'} = -\left(\frac{\delta}{\delta \psi_q}, C \frac{\delta}{\delta \bar{\psi}_{q'}} \right) \tag{31}$$

Then

$$\left(\mu_C * \prod_{q \in Q} V_q \right) (\bar{\psi}, \psi) = \left[\mathrm{e}^{\Delta} \prod_{q \in Q} V_q(\bar{\psi}_q, \psi_q) \right]_{\substack{\bar{\psi}_q = \bar{\psi} \\ \psi_q = \psi}\ \forall q \in \mathbb{N}_p} \tag{32}$$

Proof: See Appendix B. ■

Because the exponential of the Laplacian acts on a product over $q \in Q$, it suffices to get an expression for the connected part of e^{Δ}:

$$\langle V_1; \ldots; V_p \rangle = (\mathrm{e}^{\Delta})_{\mathrm{c}} \prod_{q=1}^{p} V_q \tag{33}$$

3.3 A direct resummation

We now discuss a representation of the connected part of the Laplacian as a sum over trees which corresponds to a direct resummation of the Feynman graph expansion, to motivate the solution to the problem.

Because all $\Delta_{qq'}$ commute with one another,

$$\begin{aligned} \mathrm{e}^{\Delta} &= \prod_{q=1}^{p} \mathrm{e}^{\Delta_{qq}} \prod_{q<q'} (1 + \mathrm{e}^{\Delta_{qq'} + \Delta_{q'q}} - 1) \\ &= \prod_{q=1}^{p} \mathrm{e}^{\Delta_{qq}} \sum_{G \in \mathcal{G}(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} (\mathrm{e}^{\Delta_{qq'} + \Delta_{q'q}} - 1) \end{aligned} \tag{34}$$

with G summed over all graphs on $\mathbb{N}_p$ (that is, the set of all subsets of $\mathbb{N}_p$ that have two elements). Decomposing every G into its connected components, we get (cf. (21))

$$(\mathrm{e}^{\Delta})_{\mathrm{c}} = \prod_{q=1}^{p} \mathrm{e}^{\Delta_{qq}} \sum_{G \in \mathcal{G}_{\mathrm{c}}(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} (\mathrm{e}^{\Delta_{qq'}+\Delta_{q'q}} - 1) \tag{35}$$

with G now summed over connected graphs on $\mathbb{N}_p$ [8, 10].

Applying $(\mathrm{e}^{\Delta})_{\mathrm{c}}$ to $\prod V_q$ generates the expansion of $\langle V; \ldots; V \rangle$ as a sum of values of Feynman graphs. Because every G is connected, all Feynman graphs that contribute are connected (see Sections 2.3 and 2.4 of [10]).

A term by term estimation of the sum in (35) cannot lead to convergence: $|\mathcal{G}(\mathbb{N}_p)| = 2^{\binom{p}{2}}$ and $\mathcal{G}_c$ is of similar size. The $1/p!$ in (26) decreases more slowly, so the majorant series obtained by term–by–term estimation diverges.

However, one can partly resum the expansion, to get a sum over trees (connected graphs without loops), using

Lemma 4 *To every tree $T \in \mathcal{T}(\mathbb{N}_p)$ there is a graph $H^*(T) \in \mathcal{G}(\mathbb{N}_p)$ such that $T \cap H^*(T) = \emptyset$ and $\mathcal{G}_c(\mathbb{N}_p)$ is the disjoint union*

$$\mathcal{G}_c(\mathbb{N}_p) = \bigcup_{T \in \mathcal{T}} \{H \cup T : H \subset H^*(T)\}. \tag{36}$$

We include Penrose's proof [11] in Appendix C.1. Another proof can be found in [8]. An immediate consequence of Lemma 4 is a representation of the connected correlations as a sum over trees.

Theorem 2 *Let* $\Delta^{(T)} = \sum\limits_{q=1}^{p} \Delta_{qq} + \sum\limits_{\{q,q'\} \in H^*(T)} (\Delta_{qq'} + \Delta_{q'q})$. *Then*

$$(\mathrm{e}^{\Delta})_{\mathrm{c}} = \sum_{T \in \mathcal{T}(\mathbb{N}_p)} \mathrm{e}^{\Delta^{(T)}} \prod_{\{q,q'\} \in T} (\mathrm{e}^{\Delta_{qq'}+\Delta_{q'q}} - 1). \tag{37}$$

Proof: Let $a_{qq'} = \mathrm{e}^{\Delta_{qq'}+\Delta_{q'q}} - 1$. By Lemma 4 and the binomial theorem,

$$\begin{aligned} \sum_{G \in \mathcal{G}_c(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} a_{qq'} &= \sum_{T \in \mathcal{T}(\mathbb{N}_p)} \prod_{\{q,q'\} \in T} a_{qq'} \sum_{H \subset H^*(T)} \prod_{\{q,q'\} \in H} a_{qq'} \\ &= \sum_{T \in \mathcal{T}(\mathbb{N}_p)} \prod_{\{q,q'\} \in T} a_{qq'} \prod_{\{q,q'\} \in H^*(T)} \mathrm{e}^{\Delta_{qq'}+\Delta_{q'q}} \end{aligned} \tag{38}$$

■

Finally, we can use $\mathrm{e}^{\Delta} - 1 = \Delta \int_0^1 \mathrm{d}s\, \mathrm{e}^{s\Delta}$ on every line of the tree, to get

$$(\mathrm{e}^{\Delta})_{\mathrm{c}} = \sum_{T \in \mathcal{T}(\mathbb{N}_p)} \prod_{\{q,q'\} \in T} (\Delta_{qq'} + \Delta_{q'q}) \int \mathrm{d}\mathbf{s}\, \mathrm{e}^{\Delta^{(T,\mathbf{s})}} \tag{39}$$

with $\mathbf{s} = (s_\ell)_{\ell \in T}$, $\mathrm{d}\mathbf{s} = \prod_{\ell \in T} \mathrm{d}s_\ell$, and

$$\Delta^{(T,\mathbf{s})} = \Delta^{(T)} + \sum_{\{q,q'\} \in T} s_{\{q,q'\}} (\Delta_{qq'} + \Delta_{q'q}). \tag{40}$$

By Cayley's theorem,

$$|\mathcal{T}(\mathbb{N}_p)| \le p^{p-2} \le (p-1)!\, \mathrm{e}^{p-1}, \tag{41}$$

and the $(p-1)!$ gets cancelled by the $p!$ in the denominator in (26). Thus this resummation will lead to a convergence proof if the action of each summand on $\prod V_q$ can be bounded uniformly in T. It is at this point that a problem arises with the representation (39). The action of $\mathrm{e}^{\Delta^{(T,\mathbf{s})}}$ on a monomial gives a determinant, but we have no bound for the corresponding Gram constant that is uniform in T. We explain the reasons for this in the following and then derive a representation that looks very similar to (39), but leads to uniform Gram constants.

3.4 Positivity and Gram estimates

A general feature of tree expansions like (39) is that the Laplacians appearing in $(\mathrm{e}^{\Delta})_{\mathrm{c}}$ depend on the tree T and further parameters. To discuss this dependence, we introduce the following notation. For a matrix $M \in M_p(\mathbb{C})$ and $Q \subset \mathbb{N}_p$, let

$$\Delta_Q[M] = \sum_{q,q' \in Q} M_{qq'} \Delta_{qq'}. \tag{42}$$

We abbreviate $\Delta_{\mathbb{N}_p}[M] = \Delta[M]$. The matrix element $M_{qq'}$ can be thought of as a weight factor associated to the directed line (q, q'). The matrices M occurring in our Laplacians will always be real and symmetric. The Laplacian acting in (32) is $\Delta[P]$, where $P_{qq'} = 1$ for all q and q'. Note that P is p times the orthogonal projection to the space spanned by the vector $(1, \ldots, 1)$, so P is positive (*we call a matrix M positive, and write $M \ge 0$, if M is hermitian and has nonnegative eigenvalues*). It is the positivity of the coefficient matrix M which will be crucial for good estimates. The structure of the matrices M belonging to the Laplacians in (39) is discussed in detail in Appendix C.2.

The product $V_1(\psi_1)\dots V_p(\psi_p)$ is linear in every factor, so we can for the following restrict to a single summand $v_{\overline{m}_q,m_q}$ from the representation (3) for every q. Thus the Laplacian now acts on an element of degree $\overline{m}_1+\dots+\overline{m}_p$ in $\bar\psi$ and $m_1+\dots+m_p$ in the ψ. It will be convenient to keep the coefficient function $v_{\overline{m}_q,m_q}$ and the integral over the X variables.

Let $\mathbb{B} = \mathbb{N}_p \times \mathbb{X}$, and for $\xi = (q,X) \in \mathbb{B}$ let $\Psi(\xi) = \psi_q(X)$ and $\overline{\Psi}(\xi) = \bar\psi_q(X)$. Introducing

$$\Gamma((q,X),(q',X')) = M_{qq'}C(X,X'), \tag{43}$$

and using the notation $\int_{\mathbb{B}} \mathrm{d}\xi F(\xi) = \sum_{q=1}^p \int_{\mathbb{X}} \mathrm{d}X F(q,X)$, we have

$$\Delta[M] = -\int_{\mathbb{B}} \mathrm{d}\xi \int_{\mathbb{B}} \mathrm{d}\xi' \, \frac{\delta}{\delta\Psi(\xi)}\Gamma(\xi,\xi')\frac{\delta}{\delta\overline{\Psi}(\xi')} = \mathbf{\Delta}_\Gamma. \tag{44}$$

Then

$$\mathrm{e}^{\Delta[M]} \prod_{q=1}^p {\psi_q}^{m_q}(\underline{X}_q) \prod_{q=1}^p \bar\psi_q^{\overline{m}_q}(\underline{X}'_q) = \mathrm{e}^{\mathbf{\Delta}_\Gamma} \prod_{\xi\in D} \Psi(\xi) \prod_{\xi'\in\bar D} \overline{\Psi}(\xi') \tag{45}$$

where $D \subset \mathbb{B}$ and $\bar D \subset \mathbb{B}$ are determined by the $\underline{X}_q$ and $\underline{X'}_q$.

For subsets $A, \bar A$ of $\mathbb{B}$ with $|\bar A| = |A| = d$, denote the corresponding minor of Γ by $\Gamma_{\bar A,A}$, that is, if we order $\mathbb{B}$ in some way, and if $A = \{a_1,\dots,a_d\}$ with $a_1 < \dots < a_d$ and $\bar A = \{\bar a_1,\dots,\bar a_d\}$ with $\bar a_1 < \dots < \bar a_d$, then $\Gamma_{\bar A,A}$ is the $d\times d$ matrix with entries

$$\left(\Gamma_{\bar A,A}\right)_{i,j} = \Gamma_{\bar a_i,a_j}. \tag{46}$$

Lemma 5 *There are* $\varepsilon^{A\bar A}_{D\bar D} \in \{1,-1\}$ *such that*

$$\begin{aligned} &\mathrm{e}^{\mathbf{\Delta}_\Gamma} \prod_{\xi\in D} \Psi(\xi) \prod_{\xi'\in\bar D} \overline{\Psi}(\xi') = \\ &\sum_{\substack{A\subset D,\bar A\subset\bar D\\ |A|=|\bar A|}} \varepsilon^{A\bar A}_{D\bar D} \, \det\left(\Gamma_{A,\bar A}\right) \prod_{\xi\in D\setminus A} \Psi_\xi \prod_{\xi'\in\bar D\setminus\bar A} \overline{\Psi}_\xi. \end{aligned} \tag{47}$$

Proof: Expand and permute. ■

Thus we have to estimate determinants. The point is now that good Gram estimates require some positivity.

We call a matrix A a *Gram matrix with Gram constant* α if there is a Hilbert space $\mathcal{H}$ and there are vectors f_i and g_j with $\|f_i\| \le \alpha$ and $\|g_j\| \le \alpha$ such that $A_{ij} = \langle f_i, g_j\rangle$.

Lemma 6 *If A is a Gram matrix with Gram constant α, then every minor $A_{\bar{D},D}$ is a Gram matrix with Gram constant α, and*

$$\left|\det A_{\bar{D},D}\right| \leq \alpha^{|D|+|\bar{D}|} = \alpha^{2|D|}. \tag{48}$$

If A and B are Gram matrices with Gram constants α and β, and if $C_{ij} = A_{ij}B_{ij}$, then C is a Gram matrix with Gram constant $\alpha\beta$.

Proof: The statement about minors is trivial; (48) is Gram's inequality (see e.g. Appendix B.4 of [10]). If $A_{ij} = \langle a_i, \tilde{a}_j\rangle$ and $B_{ij} = \langle b_i, \tilde{b}_j\rangle$, then $C_{ij} = \langle a_i \otimes b_i, \tilde{a}_j \otimes \tilde{b}_j\rangle$ is also a Gram matrix, with Gram constant $\alpha\beta$. ∎

Every nonnegative matrix is a Gram matrix:

Lemma 7 *Let A be a real matrix, $A = A^T$, $A \geq 0$ (that is, all eigenvalues of A are nonnegative). Then A is a Gram matrix and*

$$0 \leq \det A \leq \prod_{i=1}^{n} A_{ii}. \tag{49}$$

Proof: All eigenvalues of A are nonnegative, so there is a real matrix B, with $B = B^T \geq 0$ such that $A = BB^T = B^2$. If $b_i = (b_{ik})_k$ is the i^{th} row vector of B, this means $A_{ij} = \langle b_i, b_j\rangle$, thus in particular $A_{ii} = \|b_i\|^2$. The Gram inequality implies $\det A \leq \prod_i \|b_i\|^2$, so (49) holds. ∎

If $A = A^T$, but A is not necessarily positive, A can also be written as a Gram matrix by the polar decomposition. However, now $A_{ij} = \langle \tilde{b}_i, b_j\rangle$, and instead of an equality, one only has $A_{ii} = \langle \tilde{b}_i, b_i\rangle \leq \|\tilde{b}_i\|\|b_i\|$, so the Gram bound for the determinant is not just a bound by the product of the diagonal elements. In general, it is not easy to get bounds on the norm of the b_i and $\tilde{b}_i$. The absence of these bounds is exactly the problem with the tree representation (39) of the connected correlations (we discuss this in Appendix C.2).

But Lemma 7 also suggests a way out of this problem. The classic Brydges–Battle–Federbush interpolation that preserves stability of potentials will, as we shall see, also preserve the positivity of the matrix M that appears in $\Delta[M]$. The following immediate consequence of Lemma 7 then implies uniformity of the Gram constant.

Lemma 8 *Let M be real and symmetric, and $M \geq 0$, with diagonal elements $M_{qq} \leq 1$ for all $q \in \mathbb{N}_p$. Assume* (5)*. Then Γ, given by* (43)*, is a Gram matrix with Gram constant γ_C.*

Proof: By Lemma 7, M is a Gram matrix with Gram constant 1. Let $M_{qq'} = \langle b_q, b_{q'} \rangle$ be its Gram representation. By (5),

$$\Gamma((q,X),(q',X')) = \langle b_q \otimes f_X, b_{q'} \otimes g_{X'} \rangle. \tag{50}$$

As in the proof of Lemma 6, the Gram bound implies the statement. ■

The matrix P appearing in the Laplacian in (32) is a positive multiple of a projection, so $P \geq 0$. Moreover, all diagonal elements of P are equal to 1. Decoupling off–diagonal blocks preserves these properties:

Lemma 9 *For $M \in M_p(\mathbb{R})$, $s \in [0,1]$, and $A \subset \mathbb{N}_p$, let $\left(M^{(A,s)}\right)_{q,q'} = sM_{qq'}$ if $q \notin A$ and $q' \in A$, or if $q \in A$ and $q' \notin A$, and $\left(M^{(A,s)}\right)_{q,q'} = M_{qq'}$ otherwise. Then the diagonal elements of $M^{(A,s)}$ remain unchanged,*

$$\left(M^{(A,s)}\right)_{qq} = M_{qq} \quad \forall q \in \mathbb{N}_p, \tag{51}$$

and if $M = M^T \geq 0$, then the same holds for $M^{(A,s)}$.

Proof: It is obvious that the diagonal elements remain unchanged and that the matrix remains symmetric. By permuting the rows and columns of M with the same permutation, which amounts to a change of basis and therefore does not change positivity properties, we can assume that $A = \mathbb{N}_r$ for some $r \leq p$, and thus get, with $A^c = \mathbb{N}_p \setminus A$,

$$M^{(A,s)} = \begin{pmatrix} M_{AA} & sM_{AA^c} \\ sM^T_{AA^c} & M_{A^cA^c} \end{pmatrix} = sM + (1-s) \begin{pmatrix} M_{AA} & 0 \\ 0 & M_{A^cA^c} \end{pmatrix}. \tag{52}$$

The blockdiagonal matrix inherits positivity from M. Thus $M^{(A,s)}$ is a convex combination of two positive matrices, hence positive. ■

A tree expansion leading to uniform Gram constants is given in the following theorem.

Theorem 3 *Let M be a real symmetric matrix, and $M \geq 0$. Then*

$$\begin{aligned} \left(\mathrm{e}^{\Delta[M]}\right)_{\mathrm{c}}(\mathbb{N}_p) &= \sum_{T \in \mathcal{T}(\mathbb{N}_p)} \prod_{\{q,q'\} \in T} M_{qq'}(\Delta_{qq'} + \Delta_{q'q}) \\ &\qquad \int\limits_{[0,1]^{p-1}} \mathrm{d}\mathbf{s} \sum_{\pi \in \Pi(T)} \varphi(T,\pi,\mathbf{s}) \mathrm{e}^{\Delta[M(T,\pi,\mathbf{s})]} \end{aligned} \tag{53}$$

where $\mathbf{s} = (s_1, \ldots, s_{p-1})$, $\mathrm{d}\mathbf{s} = \mathrm{d}s_1 \ldots \mathrm{d}s_{p-1}$, $\varphi(\pi, \mathbf{s}) \geq 0$, *and* $M(T, \pi, \mathbf{s})$ *is a nonnegative symmetric matrix with diagonal entries* $(M(T, \pi, \mathbf{s}))_{qq} = M_{qq}$. *The sum over* π *runs over a* T*–dependent set* $\Pi(T)$ *of permutations* π *of* $\mathbb{N}_p$, *and*

$$\int \mathrm{d}\mathbf{s} \sum_{\pi \in \Pi(T)} \varphi(T, \pi, \mathbf{s}) = 1. \tag{54}$$

This is a variant of the BBF formula [14, 9, 15, 13]. It is proven by a repeated application of Lemma 9. We include a simple proof of Theorem 3, which avoids all explicit details about $\varphi(T, \pi, \mathbf{s})$ that we are not going to need, in Appendix A. The essential points we need, namely the positivity of $M(T, \pi, \mathbf{s})$ and (54), do not depend on these details.

3.5 Proof of Theorem 1

By Theorem 3 and Lemma 8, we can now bound $\|\langle V; \ldots; V\rangle\|_h$ essentially by a sum over trees, to which standard procedures apply, as follows. The action of $\prod_{\{q,q'\} \in T} (\Delta_{qq'} + \Delta_{q'q})$ on the homogeneous polynomial

$$\int \prod_{q=1}^{p} \mathrm{d}\underline{X}_q \mathrm{d}\underline{X}'_q \; v_{\overline{m}_q, m_q}(\underline{X}_q, \underline{X}'_q) \bar{\psi}_q^{\overline{m}_q}(\underline{X}_q) \psi_q^{m_q}(\underline{X}'_q) \tag{55}$$

is as follows. Let the tree T have incidence numbers $d_1, \ldots d_p$. Then $d_q = \theta_q + \bar{\theta}_q$ derivatives act on the q^{th} factor, θ_q of them with respect to ψ_q, and $\bar{\theta}_q$ with respect to $\bar{\psi}_q$. Because the coefficient function is totally antisymmetric, these derivatives gives rise to a combinatorial factor $\overline{m}_q(\overline{m}_q - 1) \ldots (\overline{m}_q - \bar{\theta}_q + 1)\; m_q(m_q - 1) \ldots (m_q - \theta_q + 1)$, that is,

$$\binom{\overline{m}_q}{\bar{\theta}_q} \bar{\theta}_q! \, \binom{m_q}{\theta_q} \theta_q!, \tag{56}$$

times a monomial of total degree $m_q - \theta_q + \overline{m}_q - \bar{\theta}_q$ for every q. Applying $\mathrm{e}^{\Delta[M(T,\pi,\mathbf{s})]}$ to the product of these monomials gives, by Lemma 5, a sum over subsets $A, \bar{A}$ of determinants of minors determined by A and $\bar{A}$ (these subsets are unions of subsets $\bar{A}_q$ and A_q for every factor belonging to $q \in \mathbb{N}_p$). Estimate the determinants. Because $M(T, \pi, \mathbf{s})$ is positive and has diagonal elements bounded by 1, the Gram constant of the corresponding matrices Γ is γ_C independent of $T, \mathbf{s}$, and π. Thus, by Lemma 8, each determinant is bounded by ${\gamma_C}^{a_q + \bar{a}_q}$, where $a_q = |A_q|$, $\bar{a}_q = |\bar{A}_q|$. We use (54) to do the $\mathbf{s}$–integral and the sum over π. By Cayley's theorem on the number of trees

with fixed incidence numbers $d_1, \dots, d_p$ (see, e.g. [15], Section 20.3), we can sum over incidence numbers, and are left with

$$
\begin{aligned}
\|W_p(V)\|_h \quad \le \quad & \sum_{m_1,\dots,m_p \ge 1} \sum_{\overline{m}_1,\dots,\overline{m}_p \ge 1} \mathcal{S}\left((m_q, \overline{m}_q)_{q\in\mathbb{N}_p}\right) \qquad (57) \\
& \sum_{\substack{d_1,\dots d_p \ge 1 \\ d_1+\dots+d_p = 2(p-1)}} \frac{(p-2)!}{(d_1-1)!\dots(d_p-1)!} \sum_{\substack{\theta_1,\dots,\theta_p,\bar\theta_1,\dots,\bar\theta_p \ge 0 \\ \theta_q+\bar\theta_q = d_q, \theta_q \le m_q, \bar\theta_q \le \overline{m}_q}} \binom{\overline{m}_q}{\bar\theta_q} \bar\theta_q! \binom{m_q}{\theta_q} \theta_q! \\
& \sum_{\substack{a_1,\dots,a_p \ge 0 \\ \bar a_1,\dots,\bar a_p \ge 0}} \prod_{q=1}^{p} \binom{m_q - \theta_q}{a_q} \binom{\overline{m}_q - \bar\theta_q}{\bar a_q} h^{m_q - \theta_q - a_q + \overline{m}_q - \bar\theta_q - \bar a_q} {\gamma_C}^{a_q + \bar a_q}.
\end{aligned}
$$

The binomials come from the number of subsets A_q with $|A_q| = a_q$, and

$$
\begin{aligned}
\mathcal{S} \quad = \quad & \sup_{T \in \mathcal{T}(\mathbb{N}_p)} \sup_{\tilde X} \max_i \int \mathrm{d}\underline{X} \mathrm{d}\underline{X}' \int \mathrm{d}\underline{Y} \mathrm{d}\underline{Y}' \; \delta_{\mathbb{X}}(\tilde X, Z_i) \\
& \prod_{\{q,q'\} \in T} \mathbf{C}(X_q, X'_{q'}) \prod_{q=1}^{p} |v^{(q)}_{\overline{m}_q, m_q}(\underline{X}, \underline{Y}, \underline{X}', \underline{Y}')| \qquad (58)
\end{aligned}
$$

with

$$
\mathbf{C}(X, X') = \max\{|C(X, X')|, |C(X', X)|\}, \tag{59}
$$

and where Z_i denotes one of the coordinates in $\underline{X}, \underline{Y}, \underline{X}', \underline{Y}'$ that is fixed to $\tilde X$ by the delta function. The supremum over $\tilde X$ is the supremum in the definition (8) of $\|\cdot\|_h$. (here we used that in the seminorm $|\cdot|_h$, the field–independent term is left out. We give a bound for this term in Section 3.6). Root the tree at the q for which $\tilde X$ appears as an argument of $v^{(q)}$, and perform the integrals in (58) by trimming the tree in the usual way (see [9], Appendix C) and using the summability (9) of the propagator. This gives

$$
\mathcal{S} \le |C|^{p-1} \prod_{q=1}^{p} |v^{(q)}_{\overline{m}_q, m_q}| \le {\omega_C}^{p-1} \; {\gamma_C}^{2(p-1)} \prod_{q=1}^{p} |v^{(q)}_{\overline{m}_q, m_q}|. \tag{60}
$$

The sums over a_q and $\bar a_q$ give $(h + \gamma_C)^{\overline{m}_q - \bar\theta_q + m_q - \theta_q}$. The incidence numbers d_q on the tree satisfy

$$
2(p-1) = \sum_{q=1}^{p} d_q = \sum_{q=1}^{p} (\theta_q + \bar\theta_q). \tag{61}
$$

Because $\bar\theta_q + \theta_q = d_q \ge 1$,

$$
\frac{\theta_q! \bar\theta_q!}{(d_q - 1)!} \le \max\{\theta_q, \bar\theta_q\}. \tag{62}
$$

Using this bound we can sum over the d_q (dropping the constraint that $d_1 + \ldots + d_p = 2(p-1)$) and thus remove the constraint $\bar\theta_q + \theta_q = d_q$ in the θ sums. The remaining sums over the θ_q and $\bar\theta_q$ are bounded by

$$\sum_{\theta \geq 0} \binom{m}{\theta} (h + \gamma_C)^{m-\theta} \ \max\{1, \theta\} {\gamma_C}^{\theta} \tag{63}$$

Using $\max\{1, \theta\} \leq 2^\theta$, we can bound this sum by $(h + 3\gamma_C)^m$. Thus the sum over the m_q and $\overline{m}_q$ gives

$$\|W_p(V)\|_h \leq (p-2)! \ {\omega_C}^{p-1} \ {\|V\|_{h'}}^p \tag{64}$$

with $h' = h + 3\gamma_C$. The $1/p!$ in the denominator in (26) cancels the $(p-2)!$, and (16) follows by summation over $p \geq P+1 \geq 2$. Similarly, to prove (15), we use (64), bound $(p-2)!/p! \leq 1/p$ for $p \geq 2$ and note that $\|W_1(V)\|_h = \|\mu_C * V\|_h$, so that the bound for the term $p = 1$ follows by (13) and monotonicity of $\| \cdot \|_h$ in h. Thus, summing over $p \geq 1$ gives

$$\|W(V)\|_h \leq \sum_{p \geq 1} \frac{1}{p} \, {\omega_C}^{p-1} \, {\|V\|_{h'}}^p = -\frac{1}{\omega_C} \log \left(1 - (\omega_C \|V\|_{h'})^p\right). \tag{65}$$

Finally, we prove that h' can be replaced by $h'' = h + 2\gamma_C$ if ω_C is replaced by $2\omega_C$ in the bounds in Theorem 1. In the sum over incidence numbers $d_1, \ldots, d_p$, there is the constraint $d_1 + \ldots + d_p = 2(p-1)$. We thus write

$$\frac{1}{(d_1 - 1)! \ldots (d_p - 1)!} = \frac{d_1 \ldots d_p}{d_1! \ldots d_p!} \tag{66}$$

and use the arithmetic–geometric inequality, to get

$$d_1 \ldots d_p \leq \left(\tfrac{d_1 + \ldots + d_p}{p}\right)^p = 2^p \left(1 - \tfrac{1}{p}\right)^p \leq 2^p \frac{1}{\mathrm{e}} \leq 2^{p-1}. \tag{67}$$

Then the factors $\max\{1, \theta\}$ drop out of (63), so the sums over θ_q and $\bar\theta_q$ give $2^{m_q + \overline{m}_q}$ instead of $3^{m_q + \overline{m}_q}$, and hence

$$\|W_p(V)\|_h \leq (p-2)! \ (2\omega_C)^{p-1} \ {\|V\|_{h+2\gamma_C}}^p. \tag{68}$$

3.6 Exponential decay and cumulant bounds

Let $d(X, X')$ be a pseudometric on $\mathbb{X}$ (i.e. satisfy all properties of a metric except possibly that $d(X, X') = 0$ implies $X = X'$). A typical example of this situation is if $\mathbb{X} = \mathcal{M} \times A$ with $\mathcal{M}$ a metric space, such as a torus in real space and A a finite set (such as colour and spin indices).

Theorem 4 *Assume that C satisfies* (5) *and that there are constants $\tilde{\omega}_C$ and ℓ_C such that for all X, $X' \in \mathbb{X}$*

$$\mathbf{C}(X,X') \le {\gamma_C}^2\, \tilde{\omega}_C\, \mathrm{e}^{-d(X,X')/\ell_C}. \tag{69}$$

Let $p \ge 2$, $m_1, \ldots, m_p \ge 0$, $\overline{m}_1, \ldots, \overline{m}_p \ge 0$ such that $m_q + \overline{m}_q > 0$ is even for all $q \in \mathbb{N}_p$, let $\underline{X}_q = (X_{q,1}, \ldots, X_{q,m_q})$ and $\underline{Y}_q = (Y_{q,1}, \ldots, Y_{q,\overline{m}_q})$, and let

$$\begin{aligned} \mathcal{G}\left((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p}\right) &= \left\langle \bar{\psi}^{\overline{m}_1}(\underline{Y}_1)\psi^{m_1}(\underline{X}_1); \ldots ; \bar{\psi}^{\overline{m}_p}(\underline{Y}_p)\psi^{m_p}(\underline{X}_p)\right\rangle \\ &= \tfrac{\partial^p}{\partial\lambda_1 \ldots \partial\lambda_p} \left(\mu_C * \mathrm{e}^{\sum_q \lambda_q \bar{\psi}^{\overline{m}_q}(\underline{Y}_q)\psi^{m_q}(\underline{X}_q)} \right)\Big|_{\substack{\lambda_q = 0 \forall q \\ \psi=\bar{\psi}=0}} . \end{aligned} \tag{70}$$

Then

$$\mathcal{G}\left((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p}\right) \le (p-2)!\, \tilde{\omega}_C^{p-1}\, (3\gamma_C)^{\overline{m}+m}\, \mathrm{e}^{-\frac{1}{\ell_C}\mathcal{L}((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p})} \tag{71}$$

and

$$\mathcal{G}\left((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p}\right) \le (p-2)!\, (2\tilde{\omega}_C)^{p-1}\, (2\gamma_C)^{\overline{m}+m}\, \mathrm{e}^{-\frac{1}{\ell_C}\mathcal{L}((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p})} \tag{72}$$

with $\mathcal{L}((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p})$ defined as the minimum of

$$\min_{T\in\mathcal{T}(\mathbb{N}_p)} \min \left\{ \sum_{\{q,q'\}\in T} d(X_{q,i}, Y_{q',j}) : i \in \mathbb{N}_{m_q}, j \in \mathbb{N}_{\overline{m}_{q'}} \right\} \tag{73}$$

and

$$\min_{T\in\mathcal{T}(\mathbb{N}_p)} \min \left\{ \sum_{\{q,q'\}\in T} d(Y_{q,j}, X_{q',i}) : j \in \mathbb{N}_{\overline{m}_q}, i \in \mathbb{N}_{m_{q'}} \right\}. \tag{74}$$

Proof: If we write the monomials as

$$\begin{aligned} \psi(X_1)\ldots\psi(X_m) &= \int \mathrm{d}^m \underline{X'} \prod_{k=1}^m \delta(X_k, X'_k)\psi^m(\underline{X'}) \\ &= \int \mathrm{d}^m \underline{X'} \mathbb{A}_m \left(\prod_{k=1}^m \delta(X_k, X'_k) \right) \psi^m(\underline{X'}) \end{aligned} \tag{75}$$

with the antisymmetrization operator

$$\mathbb{A}_m F(X_1, \ldots, X_m) = \frac{1}{m!} \sum_{\pi\in\mathcal{S}_m} \varepsilon(\pi) F(X_{\pi(1)}, \ldots, X_{\pi(m)}), \tag{76}$$

$\mathcal{G}$ becomes the field–independent term of the special case of $\langle V; \ldots ; V\rangle$ where V_q consists only of the term $(\overline{m}_q, m_q)$, with coefficient function

$$v^{(q)}_{\overline{m}_q, m_q}(\underline{Y'}_q, \underline{X'}_q) = \mathbb{A}_{m_q}\left[\prod_{k=1}^{m_q} \delta(X_{q,k}, X'_k)\right] \; \mathbb{A}_{\overline{m}_q}\left[\prod_{l=1}^{\overline{m}_q} \delta(Y_{q,l}, Y'_l)\right]. \tag{77}$$

Integrating over the X' and Y' variables only removes the delta functions; in particular

$$\sup_{\underline{X}_q, \underline{Y}_q} \int \prod_{q=1}^{p} \mathrm{d}^{\overline{m}_q}\underline{Y'}_q \, \mathrm{d}^{m_q}\underline{X'}_q \left| v^{(q)}_{\overline{m}_q, m_q}(\underline{Y'}_q, \underline{X'}_q)\right| \le 1 \tag{78}$$

We now consider the contribution $\mathcal{A}_T$ of one tree $T \in \mathcal{T}(\mathbb{N}_p)$ in the sum in Theorem 3 to the absolute value of the connected correlation $\mathcal{G}$. The only differences to (57) are that

- there is no sum over m_q and $\overline{m}_q$.
- because we now consider the field–independent part ($\psi = \bar\psi = 0$), all fields are integrated over; this picks out the term $A = D$ and $\bar{A} = \bar{D}$ in Lemma 5.

The second condition implies that $\mathcal{G}$ vanishes unless $m = m_1 + \ldots + m_p = \overline{m}_1 + \ldots + \overline{m}_p = \overline{m}$, which we assume from now on. Then

$$\begin{aligned}
\mathcal{A}_T \quad \le \quad & \int \mathrm{d}\mathbf{X}\, \mathrm{d}\mathbf{Y} \prod_{\{q,q'\}\in T} \mathbf{C}(X_q, \bar{X}_{q'}) \\
& \prod_{q=1}^{p} \left| v^{(q)}_{\overline{m}_q, m_q}(\bar{X}^{(q)}, \bar{Y}^{(q)}, X^{(q)}, X^{(q)})\right| \\
& \prod_{q=1}^{p} \left[\tbinom{m_q}{\theta_q} \theta_q! \, \tbinom{\overline{m}_q}{\bar\theta_q} \bar\theta_q! \; {\gamma_C}^{\overline{m}_q - \bar\theta_q + m_q - \theta_q}\right].
\end{aligned} \tag{79}$$

Here we denoted thoses integration variables on which $\mathbf{C}$ factors depend by X, the others by Y. By definition, the lines in the tree can only connect distinct q and q'. By (69),

$$\prod_{\{q,q'\}\in T} \mathbf{C}(X_q, \bar{X}_{q'}) \le {\gamma_C}^{2(p-1)} \tilde\omega_C^{p-1} \mathrm{e}^{-\frac{1}{\ell_C} \sum_{\{q,q'\}\in T} d(X_q, X_{q'})} \tag{80}$$

By (61), the factor ${\gamma_C}^{2(p-1)}$ combines with the other powers of γ_C to ${\gamma_C}^{\overline{m}+m}$. By (77), the pseudodistances appearing in the sum are all of the form $d(X_{q,i}, Y_{q',i'})$, so

$$\sum_{\{q,q'\}\in T} d(X_q, X_{q'}) \geq \mathcal{L}\left((\underline{X}_q, \underline{Y}_q)_{q\in\mathbb{N}_p}\right). \tag{81}$$

We can now bound the integral by 1 using (78) and then sum over all trees. Again, the only dependence on the tree left is in the incidence numbers. As in the proof of Theorem 1, we get factors $2^{m+\overline{m}}\, 2^{p-1}$ or $2^{m+\overline{m}}$, depending on how we do the bounds. Thus (71) and (72) hold.

■

In [16], we use these bounds to derive estimates for norms $\|\cdot\|_{h,\ell}$ which keep track of the decay length ℓ, and also construct a superrenormalizable version of the Gross–Neveu model using these norms.

4 Discussion

We have seen that the heuristic principle of resumming a graphical expansion in terms of trees can be realized in different ways, and only one of them was suited for using Gram estimates. This nonuniqueness of the tree representation is not surprising because there is no canonical way of associating a tree with a given graph. To get the decomposition in Lemma 4, we had to introduce a particular ordering on the set of all lines to obtain a well–defined map $G \mapsto T = \Phi(G)$. The BBF interpolation expansion does not group graphs into disjoint sets associated to different trees. Instead, the parameters used for the decoupling of vertices provide tree–dependent weight factors for the graphs. In the representation (39), the interpolation parameters are associated to the lines of the tree and not to the vertices; in fact, in that approach, interpolation parameters can be avoided altogether by expanding down the Laplacian in $\mathrm{e}^{\Delta_{qq'}+\Delta_{q'q}} - 1$ piece by piece (using that the Laplacian $\Delta_{(q,X),q'} = \frac{\delta}{\delta\psi_q(X)} \int \mathrm{d}X'\, C(X,X') \frac{\delta}{\delta\bar{\psi}_{q'}(X')}$ is nilpotent). Thus, although at the moment not sufficient for proving convergence, the representation (37) may be a good way of organizing perturbation expansions in practical calculations because no interpolation integrals are needed. It is better to have the sign cancellations occur in a determinant than to have at the very end a difference of two large numbers which are almost equal.

We now discuss (our understanding of) the relation of our approach to others that have appeared recently.

The construction [5] of the Gross–Neveu model and the many–fermion system is, at least technically, rather different from the approach taken here, in that it relies on forest formulas that are more explicit and that seem more closely tied to the Feynman graph expansion than our tree representations. Positivity is also used in the technical parts of the proofs in [5].

The ring expansion invented in [4] is as simple as our approach as regards the combinatorial and technical complications in the proof. Very roughly speaking, the operator R introduced there adds layers to the Feynman graphs, and thus to the spanning trees, and the condition that $\|R\| < 1$ corresponds to our condition that $\omega_C \|V\|_h < 1$. A technical difference is that the expansion in [4] is for the *externally connected* functions whereas our proof deals directly with the connected correlation functions themselves.

There are some more essential differences. Firstly, an advantage of the representation in [4] is that $\|R\| < 1$ is, while sufficient, not necessary for the representation to be defined because the formulas involve $(1 - R)^{-1}$, whose existence only requires that 1 is not an eigenvalue of R.

Secondly, Wick ordering is used in [4] to organize the ring expansion, but it plays no role in our approach. Wick ordering would simply correspond to dropping the diagonal terms $\sum_q \Delta_{qq}$ from our Laplacians. This destroys the positivity of the matrix M even in the case of the BBF decoupling. However, the positivity can easily restored by adding and subtracting the diagonal term and applying the two Laplacians one after the other, in the same way as we did it in Appendix C.2. This merely changes the Gram constant by a factor 2. Thus in our approach, Wick ordering could also be used, but it makes the constants worse.

Thirdly, an advantage of our norm bounds over those in [4] is that they are also sharp in the limit $C \to 0$, where $W(V) \to V$, and $\gamma_C \to 0$. Our shifted norm parameter h' satisfies $h' = h + 3\gamma_C \to h$, so that in the limit of no integration ($C \to 0$), we do not lose anything in the h–behaviour. In [4], the norm parameter shifts to $h + 1$.

Because our bounds are suitable for $C \to 0$, they stay useful for $C \sim \dot{C}\,\Delta t$ even in the limit $\Delta t \to 0$, and they imply that the renormalization group differential equation (RGDE)

$$\dot{W} = \Delta_{\dot{C}} W + \frac{1}{2}\left(\frac{\delta W}{\delta \psi}, \dot{C}\frac{\delta W}{\delta \bar{\psi}}\right) \tag{82}$$

is well–defined and has a solution in a ball where $\|W\|_h$ is small enough, uniformly in $|\mathbb{X}|$. This follows simply because, by definition, the effective action $W(V)$ is the solution of the RGDE (82) with the initial condition

that W equals V at flow time $t = 0$. Of course, we have not used any differential equation techniques to prove this. In particular, our proof does not constitute a nonperturbative version of Polchinski's method [17] of proving perturbative renormalizability by integrating differential inequalities.

A Polchinski–type proof of norm bounds similar to ours would probably give the simplest and most elegant tool in fermionic constructive field theory. Unfortunately, the proof in [3], which uses differential inequalities, contains a gap. This is one of the reasons why we used a discrete technique in this paper, to prove a norm bound similar to the one in [3]. Our bound is slightly weaker: in [3], a bound for $\|W(V)\|_h$ in terms of $\|V\|_{h+\gamma_C}$, i.e., without a factor in front of the γ_C, was stated. We believe that the question if and how the gap in the proof in [3] can be fixed by a differential equation argument is related to what the optimal prefactor is. This is also why we discussed this prefactor in and after Theorem 1.

One appealing feature of our norm bounds is that every order p in the expansion of W in terms of V is bounded separately. This makes it convenient for calculating $W(V)$ to low orders in V and taking norm bounds of the remainders.

The bounds given here have natural applications in RG studies of the Gross–Neveu model [16] and the many–fermion problem.

A The decoupling expansion

For $\emptyset \neq A \subset \mathbb{N}_p$, let

$$\tilde{\Delta}_{A,q}[M] = \sum_{q' \in A} M_{q'q}(\Delta_{q'q} + \Delta_{qq'}). \tag{83}$$

Then, if $M = M^T$,

$$\Delta_Q[M^{(A,s)}] = \Delta_A[M] + \Delta_{Q\setminus A}[M] + s \sum_{q \in Q\setminus A} \tilde{\Delta}_{A,q}[M]. \tag{84}$$

In particular,

$$\begin{aligned} \Delta_Q[M^{(A,1)}] &= \Delta_Q[M], \\ \Delta_Q[M^{(A,0)}] &= \Delta_A[M] + \Delta_{Q\setminus A}[M], \end{aligned} \tag{85}$$

and for all $s \in [0,1]$ and all B that satisfy either $B \cap A = 0$ or $B \subset A$,

$$\Delta_B[M^{(A,s)}] = \Delta_B[M] \tag{86}$$

because the constraint $q, q' \in B$ in the definition of Δ_B makes off–diagonal terms of type $q \in A$, $q' \notin A$ impossible. Taylor expansion now gives

$$\mathrm{e}^{\Delta_Q[M]} = \mathrm{e}^{\Delta_A[M]} \mathrm{e}^{\Delta_{Q\setminus A}[M]} + \sum_{q \in Q\setminus A} \tilde{\Delta}_{A,q}[M] \int_0^1 \mathrm{d}s \; \mathrm{e}^{\Delta_Q[M^{(A,s)}]}. \tag{87}$$

Lemma 10 *Let $Q \subset \mathbb{N}_p$, $M = M^T \in M_p(\mathbb{R})$. For $r \geq 1$ let*

$$\mathcal{S}_r(Q) = \{\mathbf{q} = (q_1, \ldots, q_r) : q_1 = \min Q, \forall i : q_i \in Q, q_i \neq q_j \text{ if } i \neq j\}. \tag{88}$$

Then for all $R \geq 1$,

$$\begin{aligned} \mathrm{e}^{\Delta_Q[M]} &= \sum_{r=1}^{R} \sum_{\mathbf{q} \in \mathcal{S}_r(Q)} \mathrm{e}^{\Delta_{Q\setminus A_r}[M]} \int \prod_{w=1}^{r-1} \mathrm{d}s_w \tilde{\Delta}_{A_w, q_{w+1}}[M_w] \; \mathrm{e}^{\Delta_{A_r}[M_r]} \\ &+ \mathcal{R}_{R+1} \end{aligned} \tag{89}$$

with $A_w = \{q_1, \ldots, q_w\}$ and the M_r defined recursively as $M_1 = M$, $M_{w+1} = {M_w}^{(A_w, s_w)}$, and a remainder term

$$\mathcal{R}_{R+1} = \sum_{\mathbf{q} \in \mathcal{S}_{R+1}(Q)} \int \prod_{w=1}^{R} \mathrm{d}s_w \tilde{\Delta}_{A_w, q_{w+1}}[M_w] \; \mathrm{e}^{\Delta_Q[M_{R+1}]}. \tag{90}$$

For all $w \in \{1,\dots,R\}$ and all $B \subset Q \setminus A_w$,

$$\Delta_B[M_w] = \Delta_B[M], \tag{91}$$

and if $M \geq 0$, then $M_w \geq 0$ for all $w \in \{1,\dots,R+1\}$.

Proof: Induction on R, with (89), (90), (91), and $M_w \geq 0$ for all $w \leq R+1$, as the inductive hypotheses. The statement for $R = 1$ is (87), with $A = \{q_1\}$. $R \mapsto R+1:$ In the remainder term, the sum over $\mathbf{q} \in \mathcal{S}_{R+1}$ includes a sum over $q_{R+1} \notin A_R$. Let $A_{R+1} = A_R \cup \{q_{R+1}\}$, and $M_{R+2} = (M_{R+1})^{(A_{R+1},s_{R+1})}$. Then $M_{R+2} \geq 0$ by Lemma 9. Now apply (87) to $\mathrm{e}^{\Delta_Q[M_{R+1}]}$. The second summand in (87) gives the new remainder term $\mathcal{R}_{R+2}$. The first summand in (87) is

$$\mathrm{e}^{\Delta_{Q\setminus A_{R+1}}[M_{R+1}]}\mathrm{e}^{\Delta_{A_{R+1}}[M_{R+1}]}. \tag{92}$$

Because $A_R \subset A_{R+1}$, $B = Q \setminus A_{R+1} \subset Q \setminus A_R$, so $B \cap A_R = \emptyset$. Thus by (86),

$$\Delta_{Q\setminus A_{R+1}}[M_{R+1}] = \Delta_B[M_R^{(A_R,s_R)}] = \Delta_B[M_R]. \tag{93}$$

By the inductive hypothesis (91), $\Delta_B[M_R] = \Delta_B[M]$, hence does not depend on $\mathbf{s}$, so its exponential can be taken out of the integral. ■

If $R = |Q|$, $\mathcal{S}_{R+1}(Q) = \emptyset$, so the remainder term vanishes, and we get

$$\mathrm{e}^{\Delta_Q[M]} = \sum_{\substack{J\subset Q\\ J\ni \min Q}} \mathrm{e}^{\Delta_{Q\setminus J}[M]}\mathcal{K}(J) \tag{94}$$

where for $|J| = j$,

$$\mathcal{K}(J) = \sum_{\mathbf{q}\in\mathcal{S}_j(J)} \int \prod_{i=1}^{j-1} \mathrm{d}s_i \tilde{\Delta}_{\{q_1,\dots,q_i\},q_{i+1}}[M_i]\, \mathrm{e}^{\Delta_J[M_j]}. \tag{95}$$

By Lemma 1, $(\mathrm{e}^{\Delta})_{\mathrm{c}}(Q) = \mathcal{K}(Q)$. It remains to bring $\mathcal{K}(\mathbb{N}_p)$ to the form stated in Theorem 3 and to show (54). The conditions in the sum over sequences in $\mathcal{S}_p(\mathbb{N}_p)$ imply that $\mathcal{S}_p(\mathbb{N}_p)$ is the set of all permutations $i \mapsto q_i = \pi(i)$ with $\pi(1) = 1$. The sum in the definition of $\tilde{\Delta}_{A,q_i}$ runs over $q_{v(i)}$ with $v(i) < i$. Thus

$$(\mathrm{e}^{\Delta[M]})_{\mathrm{c}}(\mathbb{N}_p) = \sum_{\substack{v:\{2,\dots,p\}\to\{1,\dots,p-1\}\\ v(i)<i}} \sum_{\substack{\pi\in\mathcal{S}_p\\ \pi(1)=1}} \int_{[0,1]^{p-1}} \mathrm{d}\mathbf{s}\, f(\pi,v,\mathbf{s})$$

$$\prod_{r=2}^{p} (\Delta_{\pi(v(r)),\pi(r)} + \Delta_{\pi(r),\pi(v(r))}) M_{\pi(v(r)),\pi(r)}\ \mathrm{e}^{\Delta[M_p]} \tag{96}$$

where $f(\pi, v, \mathbf{s}) \geq 0$ is a monomial in $\mathbf{s}$ arising from the repeated interpolation. We shall not need an explicit expression for it (it is given in [13] and needed for the explicit Gram representation of [12]).

The map v is a special case of a predecessor relation defining a tree: for every v in the above sum, $T_v = \{ \{v(i), i\} : i \in \{2, \ldots, p\} \}$ is a tree on $\mathbb{N}_p$. The map $v \mapsto T_v$ is injective, but not surjective because of the particular ordering induced by v (for instance, the tree $T = \{\{1,3\},\{2,3\}\}$ is not T_v for any v with $v(i) < i$). On the other hand, every tree on $\mathbb{N}_p$ is of the form $T_v^\pi = \{ \{\pi(v(i)), \pi(i)\} : i \in \{2, \ldots, p\} \}$ for some π and v. The Laplacian in (96) does not depend on v, and the product in (96) runs over lines of T_v^π. Thus we can reorganize the sums over v and π by a sum over trees T and a sum over v, π with the constraint that $T_v^\pi = T$. Defining

$$\varphi(T, \pi, \mathbf{s}) = \sum_{v : T_v^\pi = T} f(\pi, v, \mathbf{s}) \geq 0, \tag{97}$$

and $\Pi(T)$ as the set of permutations π for which $\pi(1) = 1$ and $T^\pi = T$, we get (53).

The proof of (54) is now as given by Battle and Federbush [14]: (53) holds for any family of commuting variables $\Delta_{qq'}$ and matrices $M_{qq'}$. Let T_0 be a fixed tree, $\varepsilon > 0$, and $M_{qq'} = \varepsilon$ if $\{q, q'\} \in T_0$, $M_{qq'} = 0$ otherwise (in particular, $M_{qq} = 0$). Set $\Delta_{qq'} = 1/2$. Then (53) implies that

$$\lim_{\varepsilon \to 0} \varepsilon^{-p+1} (\mathrm{e}^{\Delta[M]})_{\mathrm{c}}(\mathbb{N}_p) = \int \mathrm{d}\mathbf{s} \sum_{\pi \in \Pi(T_0)} \varphi(T_0, \pi, \mathbf{s}). \tag{98}$$

On the other hand, in the standard representation of the connected part by a sum over connected graphs,

$$(\mathrm{e}^{\Delta[M]})_{\mathrm{c}}(\mathbb{N}_p) = \prod_{q=1}^{p} \mathrm{e}^{M_{qq}\Delta_{qq}} \sum_{G \in \mathcal{G}_{\mathrm{c}}(\mathbb{N}_p)} \prod_{\{q,q'\} \in G} \left(\mathrm{e}^{M_{qq'}(\Delta_{qq'} + \Delta_{q'q})} - 1 \right), \tag{99}$$

the above choice for $M_{qq'}$ picks out the contribution from the tree T_0, so

$$\varepsilon^{-p+1} (\mathrm{e}^{\Delta[M]})_{\mathrm{c}}(\mathbb{N}_p) \to 1 \tag{100}$$

as $\varepsilon \to 0$.

Thus we get the tree expansion for the connected part of e^{Δ} stated in Theorem 3. If the ψ_q and $\bar{\psi}_q$ are independent fields, the Laplacian $\Delta[M(T, \pi, s]$ really depends on the permutation π. After evaluation at $\psi_q = \psi$ and $\bar{\psi}_q = \bar{\psi}$ for all q, the π dependence drops out and one gets back the fermionic analogue of the BBF representation given in [9].

B Proof of Lemma 3

Using source fields, we have

$$V_q(\bar{\psi},\psi) = \left[V_q\left(-\tfrac{\delta}{\delta\eta_q}, \tfrac{\delta}{\delta\overline{\eta}_q}\right) \mathrm{e}^{(\overline{\eta}_q,\psi)+(\bar{\psi},\eta_q)} \right]_{\eta_q=\overline{\eta}_q=0} \tag{101}$$

Integration, differentiation and evaluation at zero are all continuous operations on the finite–dimensional Grassmann algebra, hence interchangeable. The source term factors are in the even subalgebra, so no signs arise from commuting. Thus the left hand side becomes

$$\begin{aligned} &\prod_{q\in Q} V_q\left(-\tfrac{\delta}{\delta\eta_q}, \tfrac{\delta}{\delta\overline{\eta}_q}\right) \int \mathrm{d}\mu_C(\bar{\psi}',\psi') \prod_{q\in Q} \mathrm{e}^{(\overline{\eta}_q,\psi+\psi')+(\bar{\psi}+\bar{\psi}',\eta_q)} \\ = &\prod_{q\in Q} V_q\left(-\tfrac{\delta}{\delta\eta_q}, \tfrac{\delta}{\delta\overline{\eta}_q}\right) \mathrm{e}^{\sum_{q,q'\in Q}(\overline{\eta}_q,C\eta_{q'})+\sum_{q\in Q}[(\overline{\eta}_q,\psi)+(\bar{\psi},\eta_q)]} \end{aligned} \tag{102}$$

evaluated at $\eta_q = \overline{\eta}_q = 0$. Again using (101), the right hand side becomes

$$\left[\prod_{q\in Q} V_q\left(-\tfrac{\delta}{\delta\eta_q}, \tfrac{\delta}{\delta\overline{\eta}_q}\right) \mathrm{e}^{\Delta}\ \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]} \right]_{\eta=\overline{\eta}=0} \tag{103}$$

Because all $\Delta_{qq'}$ commute with one another, $\mathrm{e}^{\Delta} = \prod_{q,q'} \mathrm{e}^{\Delta_{qq'}}$. Because

$$\Delta_{qq'} \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]} = (\overline{\eta}_q, C\eta_{q'}) \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]}, \tag{104}$$

applying e^{Δ} gives

$$\mathrm{e}^{\Delta_{qq'}} \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]} = \mathrm{e}^{(\overline{\eta}_q,C\eta_{q'})} \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]}, \tag{105}$$

and therefore (103) is equal to

$$\left[\prod_{q\in Q} V_q\left(-\tfrac{\delta}{\delta\eta_q}, \tfrac{\delta}{\delta\overline{\eta}_q}\right) \mathrm{e}^{\sum_{q,q'}(\overline{\eta}_q,C\eta_{q'})} \mathrm{e}^{\sum_{r\in Q}[(\overline{\eta}_r,\psi_r)+(\bar{\psi}_r,\eta_r)]} \right]_{\eta=\overline{\eta}=0}. \tag{106}$$

If we set $\psi_q = \psi$ and $\bar{\psi}_q = \bar{\psi}$ for all q, we get the last line of (102).

C The direct resummation

C.1 Penrose's proof of Lemma 4

Define a map $\Phi : \mathcal{G}_c(\mathbb{N}_p) \to \mathcal{T}(\mathbb{N}_p)$ as follows. Let $G \in \mathcal{G}_c(\mathbb{N}_p)$. For a vertex $q \in \mathbb{N}_p \setminus \{1\}$ let l_q be the length of a shortest path connecting it to the vertex 1. This partitions $\mathbb{N}_p$ into sets A_l of vertices with distance l to 1. Delete all lines $\{q, q'\}$ for which $q \in A_l$ and $q' \in A_l$, for all $l \geq 1$. Call the resulting graph G'. Then every line of G' goes from A_k to A_{k+1} for some k. Also, G' is still connected, thus for every vertex $q \geq 2$, the set of lines reaching q, $R_q = \{\{q, q'\} \in G'\}$, is nonempty. Delete all lines of R_q except the one with the smallest q' from G'. The resulting graph is connected and has $q - 1$ lines. Thus it is a tree T. Let $\Phi(G) = T$.

For a tree T, $\Phi(T) = T$, so the map Φ is surjective. The decomposition given in Lemma 4 is the decomposition into preimages

$$\mathcal{G}_c(\mathbb{N}_p) = \bigcup_{T \in \mathcal{T}(\mathbb{N}_p)} \Phi^{-1}(\{T\}). \tag{107}$$

To get $\Phi^{-1}(\{T\}) = \{G \in \mathcal{G}_c(\mathbb{N}_p) : \Phi(G) = T\}$, one only has to reverse the above algorithm: let T be any tree. Group the vertices into sets A_l of distance l from 1 (i.e. root the tree at 1). Let $H^*(T)$ be the graph containing all the following lines: for $q \geq 2$ let θ be the unique line of T connecting q to a lower vertex q'; all lines $\{q'', q\}$ with $q'' > q'$ belong to $H^*(T)$. For $l \geq 1$, all lines q, q' with $q \in A_l$ and $q' \in A_l$ belong to $H^*(T)$.

By construction, all subsets H of $H^*(T)$ satisfy $\Phi(T \cup H) = T$, and if G is a connected graph containing any line not in $H^*(T)$, then $\Phi(G) \neq T$.

C.2 The matrix structure

In this section, we show that the matrices M associated to the direct resummation are band matrices and then provide examples where they have negative eigenvalues.

We first introduce a natural ordering on the vertex set $\mathbb{N}_p$. Let $V_0 = \{1\}$, and for $k \geq 1$ let V_k be the set of vertices with distance k from 1 (measured in steps when going over tree lines). V_1 is ordered by the usual ordering on $\mathbb{N}$. The set V_2 is ordered as follows: First, take the vertices q with $\pi(q)$ the smallest element of V_1, and order them in a similar way as we ordered V_1, etc. In the example shown in Figure 1, this means that the ordering of V_2 is $(4, 5, 7; 6, 8)$.

Recall that the lines of $H^*(T)$ are all those that connect vertices q, q' with $q \in V_k$ and $q' \in V_k$, and those that connect V_k and V_{k+1} and that are

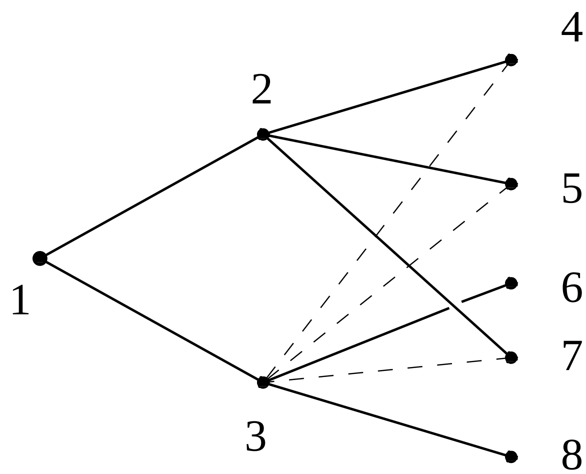

Figure 1: A tree T on $\mathbb{N}_8$ (solid lines). The dashed lines are those lines of $H^*(T)$ that connect $V_1 = \{2, 3\}$ to $V_2 = \{4, 5, 6, 7, 8\}$.

compatible with the minimality of T. The point of the above ordering is that the second condition is simple: for instance, in the example in Figure 1, the only allowed lines between V_1 and V_2 are those connecting the set $\{4, 5, 7\}$ to 3. Connecting 6 (or 8) to 2 is not allowed by construction of $H^*(T)$.

In the ordering on the vertices just introduced, the matrix $M^{(T)}$ thus takes the block form (labelled by 1, 2, 3, and the sets $W_1 = \{4, 5, 7\}$ and $W_2 = \{6, 8\}$)

$$\begin{pmatrix} 1 & s_{12} & s_{13} & 0 & 0 \\ s_{12} & 1 & 1 & \sigma_1 & 0 \\ s_{13} & 1 & 1 & 1 & \sigma_2 \\ 0 & \sigma_1^T & 1 & 1 & 1 \\ 0 & 0 & \sigma_2^T & 1 & 1 \end{pmatrix} \tag{108}$$

The blocks denoted by 1 are matrices with all entries equal to one. There are 1's in the diagonal because all lines with $q = q'$ appear and because all lines from V_k to V_k appear. The matrices σ_i comprise the s–factors from the second layer of the tree.

For general trees, the matrix $M^{(T)}$ is a block matrix of band form because the only lines allowed in $H^*(T)$ go either from V_k to V_k (diagonal blocks) or from V_k to V_{k+1} (blocks adjacent to the diagonal).

Such matrices are typically not positive: already for $p = 3$ and the tree $T = \{\{1, 2\}, \{1, 3\}\}$ (which corresponds to the left upper corner of the matrix

in (108)) and the particular values $s_{12} = 1$ and $s_{13} = 0$,

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1. \tag{109}$$

In this example, positivity is easy to repair: if all diagonal elements are replaced by 2, the matrix is just minus the one–dimensional discrete Laplacian, hence positive. Thus the matrix in (109) can be written as a difference

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{110}$$

of two Gram matrices. The first one has Gram constant $\sqrt{2}$ by Lemma 7 and the second one Gram constant 1. Upon iteration of the application of the Laplacian in two steps, the two Gram constants add up, so effectively, γ_C is replaced by $(1 + \sqrt{2})\gamma_C$. Similar tricks work for individual trees, but do not seem to yield bounds that are uniform in T.

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References

[1] K. Gawedzki and A. Kupiainen. Gross-Neveu Model Through Convergent Perturbation Expansion. *Commun. Math. Phys.*, 102:1–30, 1985.

[2] J. Feldman, J. Magnen, V. Rivasseau, and R. Sénéor. Massive Gross-Neveu Model: A Rigorous Perturbative Construction. *Phys. Rev. Lett.*, 54:1479–1481, 1985.

[3] D.C. Brydges and J.D. Wright. Mayer Expansions and the Hamilton–Jacobi Equation II. Fermions, Dimensional Reduction Formulas. J. Stat. Phys., 51:435–456 (1988); Erratum (1999), private communication, to be published

[4] J. Feldman, H. Knörrer, E. Trubowitz. A Representation for Fermionic Correlation Functions. *Commun. Math. Phys.*, 195:465–493, 1998.

[5] M. Disertori and V. Rivasseau. Continuous Constructive Fermionic Renormalization. *hep-th/9802145*, 1998, *cond-mat/9907130*, 1999.

[6] A. Abdesselam, V. Rivasseau, Trees, forests and jungles: a botanical garden for cluster expansions, in Constructive Physics, Springer, 1995

[7] A. Abdesselam, V. Rivasseau, Explicit Fermionic Tree Expansions, preprint, 1997

[8] E. Seiler. *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics.* Springer Lecture Notes in Physics. Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[9] D. C. Brydges, A Short Course on Cluster Expansions, in Critical Phenomena, random systems, gauge theories (Les Houches, Session XLIII, 1984), K. Osterwalder and R. Stora, eds., Elsevier, 1986

[10] M. Salmhofer. *Renormalization: An Introduction.* Texts and Monographs in Physics. Springer-Verlag, Berlin-Heidelberg-New York, 1999.

[11] O. Penrose, in *Statistical Mechanics*, T. Bak, ed., Benjamin, New York, 1967

[12] A. Lesniewski. Effective Action for the Yukawa$_2$ Quantum Field Theory. *Commun. Math. Phys.*, 108:437–467, 1987.

[13] G. Mack, A. Pordt. *Commun. Math. Phys.* 97: 267–298 (1985)

[14] G. Battle, P. Federbush. A phase cell cluster expansion for Euclidean field theories, *Ann. Phys.*, 142:95 (1982)

[15] J. Glimm, A. Jaffe, Quantum Physics, Second Edition, Springer, Heidelberg, 1987

[16] M. Salmhofer, C. Wieczerkowski, Construction of the Renormalized Gross-Neveu Trajectory in $2-\epsilon$ Dimensions, mp_arc 99-255, to appear

[17] J. Polchinski, *Nucl. Phys.* B 231 (1984) 269